ON-LINE TRADING AS A RENEWAL PROCESS: WAITING TIME AND INSPECTION PARADOX

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We briefly review our recent studies on stochastic-process modelling internet on-line trading. We present a way to evaluate the average waiting time between the observation of the price in financial markets and the next price change, especially in an on-line foreign exchange trading service for individual customers via the Internet. The basic method of our approach depends on the so-called renewal-reward theorem. Assuming that the stochastic process that models the price change is a renewal process, we use the theorem to calculate the average waiting time of the process. The so-called "inspection paradox" is discussed, which, in general, means that the average durations is shorter than the average waiting time.

Introduction

Financial data have attracted a lot of attention from physicists as informative material to investigate the macroscopic behavior of markets[1–3]. Some of these studies are restricted to the stochastic variables of the price changes (returns) and most of them concern a keyword: *Fat tails* of the distributions[1]. However, also the distribution of time intervals can deliver useful information on the markets and it is worth while to investigate these properties extensively[4–9] and if possible, to apply the gained knowledge to financial engineering.

Fluctuations in time intervals between events are not only peculiar to financial markets, but also very common in science. For instance, the spike train of a single neuron is characterized by a time series in which the time difference between consecutive spikes is not constant but fluctuates. This stochastic process, specified by the so-called Inter-Spike Intervals (ISI), is one of such examples[10,11]. The average of the ISI is about a few milliseconds and the distribution of the durations (intervals) is well-described by the *Gamma distribution*[11].

On the other hand, in financial markets, for instance, the time intervals of two consecutive transactions of BUND futures (BUND is the German word for bond) and BTP futures (BTP is the middle and long term Italian Government bonds with fixed interest rates) traded at LIFFE (LIFFE stands for London International Financial Futures and Options Exchange) are ~ 10 seconds and are well-fitted by the so-called *Mittag-Leffler distribution*[5–7]. The Mittag-Leffler distribution behaves as a stretched exponential distribution in the short interval regime, whereas for the long interval regime, the function has a power-law tail. Therefore, the behavior of the distribution described by the Mittag-Leffler function varies from stretched exponential to power-law at some intermediate critical interval[12].

TABLE I: Three typical examples with fluctuation between events.

	ISI	BUND future	Sony bank rate
Average duration	~3 [ms]	~10 [s]	~20 [min]
PDF	Gamma	Mittag-Leffler	Weibull

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The Sony bank USD/JPY exchange rate[13], (i.e. the rate for individual customers of the Sony bank in their on-line foreign exchange trading service) is a good example to be checked against the Mittag-Leffler distribution.

Actually, our results implied that the Mittag-Leffler distribution does not well fit the Sony bank rate[15]. The Sony bank rate has ~20 minutes[16] as the average time interval which is much longer than other market time intervals such as those for the BUND future. This is due to the fact that the the Sony back rate can be regarded as a so-called *first-passage process*[17-22] for raw market data. In Table I, we list the average time intervals and the probability distribution function (PDF) that describes the data with fluctuation between the events for typical three examples, namely, the ISI, the BUND future and the Sony bank rate. From this table, an important question arises. Namely, how long do the customers of the Sony bank should wait between observing the price and the next price change? This type of question never occurs in the case of the ISI or of the BUND future, because the average time intervals are too short to evaluate such informative measure.

For the customers, an important (relevant) quantity is the waiting time rather than the time interval between consecutive rate changes. Here, the waiting time is defined as the time the customer has to wait between the instant in which they enter the market in the World-Wide Web and the next price change[13]. If the sequence of time intervals is non-exponential and the customers observe the rate at random on the time axis, the distribution of the waiting time no longer coincides with the distribution of the time intervals.

In this review article, we present a useful way to evaluate moments of arbitrary order for the waiting time and for arbitrary duration distribution of price changes as well as observation time distribution, by directly deriving the waiting time distribution.

This paper is organized as follows. In the next section II, we introduce the Sony bank rate[13] which is generated from the high-frequency foreign exchange market rate via the rate window with width 2ε yen ($\varepsilon = 0.1$ yen for the Sony bank). Then in the subsequent section III, we explain our method to derive waiting-time distributions. We show that our treatment reproduces the result obtained by the renewal-reward theorem. We also evaluate the deviation around the average waiting time for the Weibull first-passage time distribution and uniform observation-time distribution. We find that the resultant standard deviation is the same order as the average waiting time. We test our analysis for several observation-time distributions and calculate higher-order moments. The so-called "inspection paradox"[14] is mentioned in section IV, which means in general that the average of durations is shorter than the average waiting time. The last section V contains concluding remarks.

The Sony Bank Rate: An Example of First-passage Process

The Sony bank rate we deal with in this paper is the rate for individual customers of the Sony bank[13] in their on-line foreign exchange trading service via the Internet. If the USD/JPY market rate changes by greater or equal to 0.1 yen, the Sony bank USD/JPY exchange rate is updated to the market rate. In this sense, the Sony bank rate can be regarded as a first passage processes[17–22]. In Fig. 1, we show the mechanism of generating the Sony bank rate from the market rate (this process is sometimes referred to as a *first exit process*[31]). As shown in figure 1, the time difference between two consecutive points in the

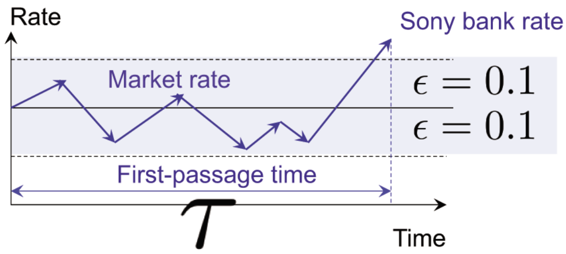

Fig. 1. An illustration of generating the ltered rate by the rate window with width 2 from the market rate. If the market rate changes by a quantity greater or equal to 0.1 yen, the Sony bank USD/JPY exchange rate is updated to the market rate.

Sony bank rate becomes longer than the time intervals of the market rates. In Table II, we show several data concerning the Sony bank USD/JPY rate vs. tick-by-tick data by Bloomberg for USD/JPY rate. It is a non-trivial problem to ask what kind of distribution is suitable to explain the distribution of the first-passage time. For this problem, we attempted to check several statistics from both the analytical and the empirical points of view under the assumption that the first-passage time might obey a Weibull distribution[23–25]. We found that the data are well fitted by

TABLE II: The Sony bank USD/JPY exchange rate vs. tick-by-tick data for USD/JPY exchange rate.

	Sony bank rate	tick-by-tick data
# of data a day	~ 70	~ 10,000
The smallest price change	0.1 yen	0.01 yen
Average duration	~ 20 minutes	~ 7 seconds

a Weibull distribution. This fact means that the difference between successive rate changes in Sony bank fluctuates and has some memory.

Derivation of the Waiting Time Distribution

In this section, we derive the distribution of the waiting time for the customers. Our approach enables us to evaluate not only the first moment of the waiting time but also moments of any order.

The probability distribution of the waiting time : We first derive the probability distribution function of the waiting time s. Let us suppose that the difference between two consecutive points of the Sony bank rate change, namely, the first-passage time τ follows the distribution with probability density function $P_W(\tau)$. Then, the customers observe the rate at time t ($0 \le t \le \tau$) that should be measured from the point at which the rate previously changed. In Fig. 2, we show the relation among these variables τ, t and s in the time axis. The waiting time for the customers is naturally defined as $s \equiv \tau - t$. Now, notice

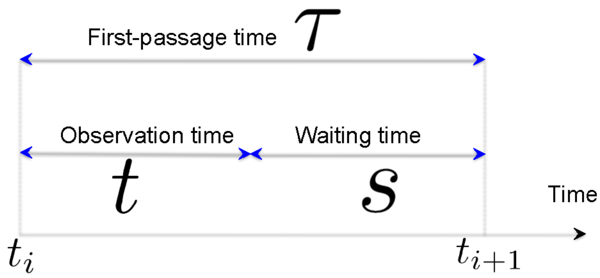

Fig. 2. The relation these points τ, t and s in time axis. The first-passage time τ is given by $\tau = t_{i+1} - t_i$. The observation time is measured from the point t_i.

that the distribution $\Omega(s)$ can be written in terms of the first-passage time distribution (with density $P_W(\tau)$) and the observation time distribution (with density $P_O(t)$) of the customers as a convolution

$$\Omega(s) \propto \int_0^\infty d\tau\, Q(s \mid \tau, t)\, P_O(t)\, P_W(\tau)\,. \qquad (1)$$

In this equation, the conditional probability density $Q(s \mid \tau, t)$ that the waiting time takes the value s provided that the observation time and the first-passage time were given as t and τ, respectively, is given by

$$Q(s \mid \tau, t) = \delta(s - \tau + t) \qquad (2)$$

where $\delta(\cdot)$ is Dirac's delta function. Taking into account the normalization constant of $\Omega(s)$, we have

$$\Omega(s) = \frac{\int_0^\infty d\tau P_W(\tau) \int_0^\tau dt\, \delta(s - \tau + t)\, P_O(t)}{\int_0^\infty ds \int_0^\infty d\tau\, P_W(\tau) \int_0^\tau d\tau\, Q(s - \tau + t)\, P_O(t)} \qquad (3)$$

where, again, t denotes the observation time for the customers. The result of the renewal-reward theorem : $w = \langle s \rangle = E(\tau^2)/2E(\tau)$ (see for example[27]) is recovered by inserting a uniformly distributed observation time distribution $P_O(t) = 1$ into the above expression. Indeed, we have

$$w = \langle s \rangle = \int_0^\infty dss\, \Omega(s) = \frac{\int_0^\infty dss \int_s^\infty d\tau P_W(\tau)}{\int_0^\infty ds \int_s^\infty d\tau P_W(\tau)}$$

$$= \frac{\int_0^\infty \frac{d}{ds}\{s^2/2\} ds \int_s^\infty d\tau P_W(\tau)}{\int_0^\infty \frac{d}{ds}\{s\} ds \int_s^\infty d\tau P_W(\tau)}$$

$$= \frac{(1/2)\int_0^\infty s^2 P_W(s)\, ds}{\int_0^\infty s P_W(s)\, ds} = \frac{E(\tau^2)}{2E(\tau)} \qquad (4)$$

where we defined the n-th moment of the first-passage time $E(\tau^n)$ by

$$E(\tau^n) = \int_0^\infty dss^n P_W(s)\,. \qquad (5)$$

More generally, we may choose a non-uniform $P_O(t)$. For this general form of the observation time distribution, the probability distribution of the waiting time s is given as follows.

$$\Omega(s) = \frac{\int_s^\infty d\tau P_W(\tau)\, P_O(\tau - s)}{\int_0^\infty ds \int_s^\infty d\tau P_W(\tau) P_O(\tau - s)}$$

$$= \frac{\int_s^\infty d\tau P_W(\tau)\, P_O(\tau - s)}{E(t) - \delta_1} \qquad (6)$$

where we defined δ_n by

$$\delta_n = \int_0^\infty \frac{dss^n}{n} \int_s^\infty P_W(\tau) \frac{\partial\, P_O(\tau - s)}{\partial s}\,. \qquad (7)$$

By using the same method as in the derivation of the distribution $\Omega(s)$, we easily obtain the first two moments of the waiting time distribution as

$$\langle s \rangle = \frac{E(\tau^2)/2 - \delta_2}{E(\tau) - \delta_1}, \ \langle s^2 \rangle = \frac{E(\tau^3)/3 - \delta_3}{E(\tau) - \delta_1} \tag{8}$$

and the study of the standard deviation leads to

$$\sigma = \sqrt{\frac{4E(\tau^3)E(\tau) - 3E(\tau^3) + G_{\delta_1,\delta_2,\delta_3}}{12\left(E(\tau) - \delta_1\right)^2}} \tag{9}$$

$$G_{\delta_1,\delta_2,\delta_3} = -4\delta_1 E(\tau^3) - 12\delta_3 E(\tau) + 12\delta_2 E(\tau^2) + 12\delta_1, \delta_3 - 12\delta_2^2 \tag{10}$$

where we defined

$$\langle s^n \rangle = \int_0^\infty dss^n \Omega(s) \,. \tag{11}$$

Thus, this probability distribution $\Omega(s)$ enables us to evaluate moments of any order for the waiting time. We consider the case of $P_O(\tau) = 1$ as an example. This case corresponds to the result obtained by the renewal-reward theorem[25]. We find that $\delta_n = 0$ holds for arbitrary integer n. Thus, the waiting time distribution $\Omega(s)$ leads to

$$\Omega(s) = \frac{\int_s^\infty P_W(\tau)}{E(\tau)} \,. \tag{12}$$

Then, the average waiting time and the deviation around the value lead to

$$w = \frac{E(\tau^2)}{2E(\tau)}, \ \sigma = \sqrt{\frac{4E(\tau^3)E(\tau) - 3E(\tau^2)^2}{12E(\tau)^2}} \tag{13}$$

For a Weibull distribution having the parameters m, a, the above results can be re-written as

$$\Omega(s) = \frac{me^{-s^m/a}}{a^{1/m}\Gamma\left(\frac{1}{m}\right)} \tag{14}$$

$$w = a^{1/m}\frac{\Gamma\left(\frac{2}{m}\right)}{\Gamma\left(\frac{1}{m}\right)} \tag{15}$$

$$\sigma = \frac{a^{1/m}\sqrt{\Gamma(1/m)\,\Gamma(3/m) - \Gamma(2/m)^2}}{\Gamma(1/m)} \tag{16}$$

where we defined the Gamma function as

$$\Gamma(x) = \int_0^\infty dt\, t^{x-1}\, e^{-t} \,. \tag{17}$$

It is important for us to notice that for an exponential distribution $m = 1$, we have $w = \sigma = a$ by taking into account the fact that $\Gamma(n) = (n-1)!$. Moreover, the average waiting time w is identical to the average time interval $E(\tau)$ since $w = E(\tau^2)/2E(\tau) = E(\tau)$ holds if and only if $m = 1$ (The rate change follows a Poisson arrival process). These results are obtained by using a different method in our previous studies[25]. In Fig. 3, we plot the distribution $\Omega(s)$ for $m = 1$, 2 and $m = 0.585$.

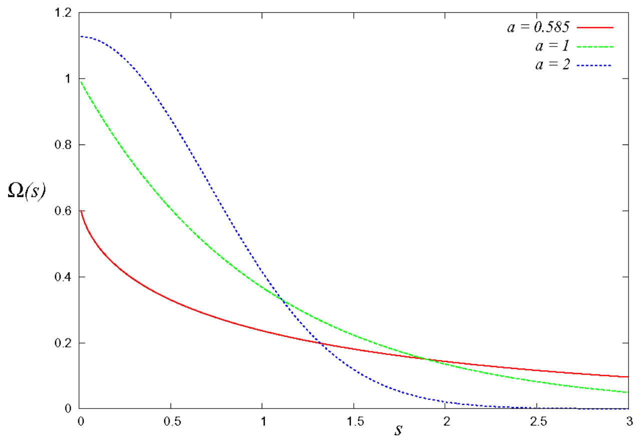

Fig. 3. The distribution of waiting time for a Weibull distributing $\Omega(s)$ with $a = 1$ and $m = 0.59$; 1 and 2.

Comparison with empirical data analysis : It is time to compare the analytical result with that of the empirical data analysis. For uniform observation time distribution $P_O(t) = 1$, we obtained $\sigma = 60.23$ minutes. On the other hand, from empirical data analysis, we evaluate the quantity (13) by sampling the moment as $E(\tau^n) = (1/N)\sum_{i=1}^N \tau_i^n$ directly from Sony bank rate data[13] and find $\sigma = 74.35$ minutes. There exists a finite gap between the theoretical prediction and the result by the empirical data analysis, however, both results are of the same order of magnitude. The gap might become smaller if we take into account the power-law tail of the first-passage time distribution. In fact, we showed that for the average waiting time, the power-law tail makes the gap between the theoretical prediction and empirical observation smaller[15].

Inspection Paradox

Here we encounter the situation which is known as "inspection paradox". For the Weibull distribution, the paradox occurs for $m < m_c = 1$. Namely, for this regime, we have $\langle s \rangle > \langle \tau \rangle$ (see Fig. 4). In general, it means that the average of durations (first-passage times) is shorter than the average waiting time. This fact is quite counter- intuitive because the customer checks the rate at a time between arbitrary consecutive rate changes. This fact is intuitively understood as follows. When the parameter m is smaller than m_c, the bias of the duration is larger than that of the exponential distribution. As a result, the chance for customers to check the rate within large intervals between consecutive price changes is more frequent than the chance they check the rate within shorter intervals. Then, the average waiting time can become longer than the average duration.

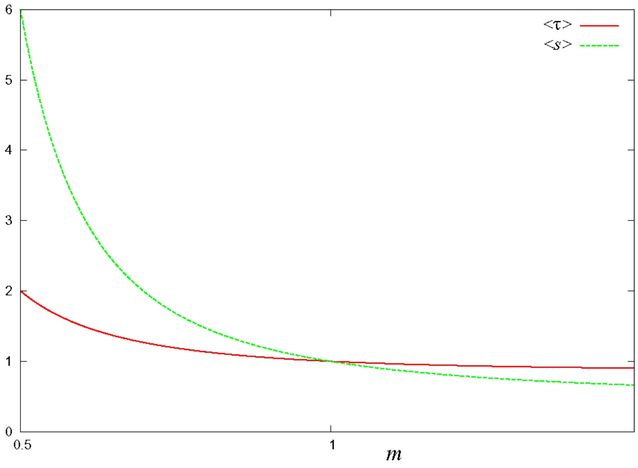

Fig. 4. Average duration $\langle\tau\rangle$ and average waiting time <s> as a function of m for a Weibull duration distribution with a = 1. The inspection paradox occurs for $m < m_c = 1$.

Concluding Remarks

As we showed in this paper, our queueing theoretical approach might be useful to build artificial markets such as the on-line trading service so as to have a suitable waiting time for the individual customers by controlling the width of the rate window. Moreover, the theoretical framework we provided here could predict the average waiting time including the deviation from empirical results.

We hope that this review article might help researchers or financial engineers when they attempt to build a suitable on-line system for their customers. □

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